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A closed-form representation for the derivative of non-symmetric tensor power series

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To the memory of our colleague and teacher Professor Yavuz Başar (1935–2002)

Abstract

In the present paper a closed-form representation for the derivative of non-symmetric tensor power series is proposed. Particular attention is focused on the special case of repeated eigenvalues. In this case, a non-symmetric tensor can possess no spectral decomposition (in diagonal form) such that the well-known solutions in terms of eigenprojections as well as basis-free representations for isotropic functions of symmetric tensor arguments cannot be used. Thus, our representation seems to be the only possibility to calculate the derivative of non-symmetric tensor power series in a closed form. Finally, this closed formula is illustrated by an example being of special importance in large strain anisotropic elasto-plasticity. As such, we consider the exponential function of the velocity gradient under simple shear. Right in this loading case the velocity gradient has a triple defective eigenvalue excluding the application of any other solutions based on the spectral decomposition.

Keywords: Non-symmetric tensor power series; Derivative of tensor function; Isotropic tensor functions; Exponential tensor function; Fourth-order tensors; Anisotropic elasto-plasticity

1. Introduction

Tensor-valued functions defined in terms of power series with respect to non-symmetric tensor arguments are commonly used in continuum mechanics. For example, the exponential function of the velocity gradient or other non-symmetric strain rates is very suitable for the formulation of evolution equations in large strain viscoplasticity (see e.g. Sansour and Kollmann, 1998) as well as in anisotropic and particularly single crystal plasticity (see e.g. Steinmann and Stein, 1996; Miehe, 1996). However, by implementing the exponential mapping algorithm one needs not only the exponential function itself but also its derivative

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which is indispensable for the formulation of consistent algorithmic tangent moduli. If symmetric, an isotropic tensor function along with its derivative can easily be obtained on the basis of the spectral decomposition (Bowen and Wang, 1970, 1971; Chadwick and Ogden, 1971). This is certainly the case for exponential and any other tensor functions expressible in terms of power series since such functions represent a subclass of isotropic tensor functions. Thereby, the derivative of the isotropic tensor function can alternatively be obtained by means of the so-called basis-free representations based on the spectral decomposition as well (Carlson and Hoger, 1986; Xiao, 1995; Xiao et al., 1998). On the contrary, a non-symmetric tensor argument can possess no spectral decomposition (in diagonal form) which necessitates to seek for other approaches (see e.g. Moler and Van Loan, 1978). A direct calculation of the derivative of tensor power series (de Souza Neto, 2001) is numerically extremely expensive and for this reason inefficient. A more elegant approach is based on the recurrent computation of the coefficients in the tensor power series representation resulting from sequential application of the Cayley–Hamilton theorem. Until now this procedure has only been used for the calculation of tensor power series (Miehe, 1996; Sansour and Kollmann, 1998) though it can likewise be implemented also for their derivative. With respect to numerical costs this procedure is more economical since the maximal tensor power is restricted to two. However, the underlying representation cannot be regarded as closed one. Indeed, the calculation of scalar coefficients imbedded in this representation is based on the series being infinite for exponential and other tensor functions defined by infinite power series.

To overcome the above mentioned difficulties we propose in the present paper a closed-form solution for the derivative of tensor power series. The underlying idea is rather simple. By means of the sequential application of the Cayley–Hamilton theorem one can obtain the closed-form representation for the derivative of tensor power series. The scalar coefficients in this representation depend only on the principal invariants or eigenvalues of the tensor argument. For this reason this representation must hold for all tensor arguments with the same principal invariants or eigenvalues independent of whether these tensors possess a spectral decomposition or not. In the case of distinct eigenvalues a tensor argument always permits a spectral decomposition. Thereby, we may use the closed-form solution in terms of the eigenvalue-bases even if the tensor argument is non-symmetric (Itskov, 2002). Using a limiting procedure this solution can then be extended to the case of repeated eigenvalues where a non-symmetric tensor argument generally possesses no spectral decomposition.

The paper is organized as follows. First, we introduce some necessary tensor notations and definitions (Section 2). A difficulty in dealing with non-symmetric tensors are partly complex eigenvalues and eigenprojections, so we are required to define complex vectors and tensors as well as operations with them (see also Boulanger and Hayes, 1993). In Section 3 we recall the well-known recurrent procedure for the calculation of tensor power series and extend it to the computation of their derivative. The recurrent computations can be avoided by means of the closed-form representation obtained in Section 4. For the derivative of the tensor power series we present in Section 5 an alternative form of the closed formula. It is based on the direct differentiation of the above mentioned tensor power series representation resulting from the application of the Cayley–Hamilton theorem. By using some universal tensor identities both results are shown to be equivalent. Particular attention is focused in Sections 4 and 5 on the special case of repeated eigenvalues. In this case a tensor argument can possess no spectral decomposition such that our solutions seem to be the only possibility to calculate the derivative of non-symmetric tensor power series in a closed form. On the contrary, in the special case of the tensor argument possessing a spectral decomposition the solutions proposed in the paper are shown in Section 6 to coincide with the well-known representations for isotropic functions of symmetric tensor arguments (see e.g. Carlson and Hoger, 1986; Xiao, 1995; Xiao et al., 1998). Finally, an application of our closed formulas is illustrated in Section 7 by an example of the exponential function of the velocity gradient under simple shear. In this loading case the velocity gradient has a triple eigenvalue and only two linearly independent eigenvectors excluding the application of any other solutions based on the spectral decomposition.

2. Basic notations and definitions

Let $\mathbb{C}^3$ be a three-dimensional vector space over the field of complex numbers $\mathbb{C}$. The scalar product of two complex vectors is defined without using the complex conjugate values and is linear and commutative with respect to both arguments (see Boulanger and Hayes, 1993):

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}, \quad \forall \boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^3. \tag{1}$$

Let $\mathbf{C}$lin be a set of all linear mappings of $\mathbb{C}^3$ into itself. The elements of $\mathbf{C}$lin are called second-order tensors (bold capitals). Second-order tensors can be formed from vectors with the aid of the tensor product "$\otimes$" defined by

$$(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{x} = (\boldsymbol{b} \cdot \boldsymbol{x})\boldsymbol{a}, \quad \boldsymbol{x}(\boldsymbol{a} \otimes \boldsymbol{b}) = (\boldsymbol{x} \cdot \boldsymbol{a})\boldsymbol{b} \qquad \forall \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{x} \in \mathbb{C}^3. \tag{2}$$

Through the standard operations of sum and multiplication with a scalar $\mathbf{C}$lin constitutes a finite-dimensional vector space. The set $\mathbf{L}$in of all linear mappings within the three-dimensional vector space $\mathbb{R}^3$ over the field of real numbers $\mathbb{R}$ represents an important subset of $\mathbf{C}$lin. Symmetric and orthogonal second-order tensors constitute in turn subsets of $\mathbf{L}$in defined in the following manner: $\mathbf{S}\mathrm{ym} = \{\mathbf{A} \in \mathbf{L}\mathrm{in} : \mathbf{A} = \mathbf{A}^{\mathrm{T}}\}$, $\mathbf{O}\mathrm{rth} = \{\mathbf{Q} \in \mathbf{L}\mathrm{in} : \mathbf{Q} = \mathbf{Q}^{-\mathrm{T}}\}$.

Fourth-order tensors form a set $\mathscr{C}$lin ($\mathscr{L}$in) of all linear mappings of $\mathbf{C}$lin ($\mathbf{L}$in) into itself such that (cf. Del Piero, 1979):

$$\mathbf{B} = \mathscr{D} : \mathbf{A}, \quad \mathbf{B} \in \mathbf{C}\mathrm{lin} \qquad \forall \mathbf{A} \in \mathbf{C}\mathrm{lin}, \ \forall \mathscr{D} \in \mathscr{C}\mathrm{lin}. \tag{3}$$

For the construction of fourth-order tensors from second-order ones we introduce the tensor products "$\otimes$ " and "$\times$" defined by (Itskov, 2000, 2002)

$$\mathbf{A} \otimes \mathbf{B} : \mathbf{C} = \mathbf{ACB}, \quad (\mathbf{A} \times \mathbf{B}) : \mathbf{C} = (\mathbf{B} : \mathbf{C})\mathbf{A} \qquad \forall \mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbf{C}\mathrm{lin}. \tag{4}$$

Further, the simple contraction of fourth- and second-order tensors can be defined in the following manner

$$(\mathbf{A}\mathscr{D}\mathbf{B}) : \mathbf{C} = \mathbf{A}(\mathscr{D} : \mathbf{C})\mathbf{B}, \quad \forall \mathscr{D} \in \mathscr{C}\mathrm{lin}, \ \forall \mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbf{C}\mathrm{lin}. \tag{5}$$

The main object of the paper are tensor-valued tensor functions $\mathbf{G}(\mathbf{A})$ defined by tensor power series

$$\mathbf{G}(\mathbf{A}) = \chi_0\mathbf{I} + \chi_1\mathbf{A} + \chi_2\mathbf{A}^2 + \chi_3\mathbf{A}^3 + \cdots \quad \forall \mathbf{A} \in \mathbf{D}\mathrm{lin} \subset \mathbf{L}\mathrm{in}, \tag{6}$$

where $\chi_i \in \mathbb{R}$ $(i = 1, 2, \ldots)$ denote scalar constants. If infinite, the power series (6) is assumed to be convergent over the definition domain $\mathbf{D}$lin of the corresponding tensor function $\mathbf{G}(\mathbf{A})$.

The tensor power series (6) represent a subclass of isotropic tensor functions characterized by the condition (see e.g. Truesdell and Noll, 1965):

$$\mathbf{G}(\mathbf{QAQ}^{\mathrm{T}}) = \mathbf{QG}(\mathbf{A})\mathbf{Q}^{\mathrm{T}}, \quad \forall \mathbf{Q} \in \mathbf{O}\mathrm{rth}. \tag{7}$$

For example, the exponential tensor function can be defined in the form (6) by

$$\exp(\mathbf{A}) = \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{A}^n. \tag{8}$$

Of special importance for the following discussion is the derivative of a scalar- $\alpha(\mathbf{A}) : \mathbf{L}\mathrm{in} \to \mathbb{R}$ and a tensor-valued function $\mathbf{G}(\mathbf{A}) : \mathbf{L}\mathrm{in} \to \mathbf{L}\mathrm{in}$ with respect to their tensor argument $\mathbf{A} \in \mathbf{L}\mathrm{in}$. These functions are said to be differentiable if the directional (Gateaux) derivatives

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\alpha(\mathbf{A} + s\mathbf{X})\right|_{s=0} \quad \text{and} \quad \left.\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{G}(\mathbf{A} + s\mathbf{X})\right|_{s=0}$$

exist in a neighbourhood of $\mathbf{A}$ and are continuous at $\mathbf{A}$ and there exist a second- $\alpha(\mathbf{A})_{,\mathbf{A}} \in \mathbf{Lin}$ or a fourth-order tensor $\mathbf{G}(\mathbf{A})_{,\mathbf{A}} \in \mathscr{L}\text{in}$, respectively, such that (see e.g. Truesdell and Noll, 1965)

$$\alpha(\mathbf{A})_{,\mathbf{A}} : \mathbf{X} = \frac{\mathrm{d}}{\mathrm{d}s}\alpha(\mathbf{A} + s\mathbf{X})\bigg|_{s=0}, \quad \mathbf{G}(\mathbf{A})_{,\mathbf{A}} : \mathbf{X} = \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{G}(\mathbf{A} + s\mathbf{X})\bigg|_{s=0} \qquad \forall \mathbf{X} \in \mathbf{Lin}. \tag{9}$$

The tensors $\alpha(\mathbf{A})_{,\mathbf{A}}$ and $\mathbf{G}(\mathbf{A})_{,\mathbf{A}}$ are referred to as derivative or gradient of the tensor functions $\alpha(\mathbf{A})$ and $\mathbf{G}(\mathbf{A})$, respectively.

By using the definitions $(4)_1$ and (9) one can easily obtain

$$\mathbf{A}_{,\mathbf{A}} = \mathscr{I}, \quad \mathbf{A}^n{}_{,\mathbf{A}} = \sum_{r=0}^{n-1} \mathbf{A}^{n-1-r} \otimes \mathbf{A}^r, \qquad n = 1, 2, \ldots, \tag{10}$$

where $\mathscr{I} = \mathbf{I} \otimes \mathbf{I}$ represents the fourth-order identity tensor. Of special importance are also the following product rules of differentiation (Itskov, 2000, 2002)

$$(\mathbf{AB})_{,\mathbf{C}} = \mathbf{A}_{,\mathbf{C}}\,\mathbf{B} + \mathbf{A}\mathbf{B}_{,\mathbf{C}}\,, \quad (\alpha\mathbf{A})_{,\mathbf{B}} = \mathbf{A} \times \alpha_{,\mathbf{B}} + \alpha\mathbf{A}_{,\mathbf{B}}\,, \tag{11}$$

directly resulting from the definitions (4), (5) and (9).

3. A recurrent calculation of tensor power series and their derivative

The tensor power series (6) and their derivative can be computed by means of the recurrent relations recalled below. The recurrent procedure is based on the sequential application of the Cayley–Hamilton theorem written as

$$\mathbf{A}^3 - \mathrm{I}_{\mathbf{A}}\mathbf{A}^2 + \mathrm{II}_{\mathbf{A}}\mathbf{A} - \mathrm{III}_{\mathbf{A}}\mathbf{I} = \mathbf{0}, \quad \forall \mathbf{A} \in \mathbf{Lin}, \tag{12}$$

where the coefficients $\mathrm{I}_{\mathbf{A}}$, $\mathrm{II}_{\mathbf{A}}$ and $\mathrm{III}_{\mathbf{A}}$ represent the principal invariants of $\mathbf{A}$ defined by

$$\mathrm{I}_{\mathbf{A}} = \mathrm{tr}\,\mathbf{A}, \quad \mathrm{II}_{\mathbf{A}} = \frac{1}{2}\left[(\mathrm{tr}\,\mathbf{A})^2 - \mathrm{tr}\,\mathbf{A}^2\right], \quad \mathrm{III}_{\mathbf{A}} = \det \mathbf{A}. \tag{13}$$

By virtue of (12) higher powers of $\mathbf{A}$ are expressible by

$$\mathbf{A}^k = c_k^{(0)}\mathbf{I} + c_k^{(1)}\mathbf{A} + c_k^{(2)}\mathbf{A}^2, \tag{14}$$

where the unknown coefficients $c_k^{(r)}$ $(r = 0, 1, 2;\ k = 0, 1, 2, \ldots)$ can be calculated by means of the following recurrent relations (see e.g. Sansour and Kollmann, 1998)

$$\begin{aligned} &c_l^{(r)} = \delta_{rl}, \quad r, l = 0, 1, 2, \\ &c_k^{(0)} = c_{k-1}^{(2)}\mathrm{III}_{\mathbf{A}}, \quad c_k^{(1)} = c_{k-1}^{(0)} - c_{k-1}^{(2)}\mathrm{II}_{\mathbf{A}}, \quad c_k^{(2)} = c_{k-1}^{(1)} + c_{k-1}^{(2)}\mathrm{I}_{\mathbf{A}}, \qquad k = 1, 2, \ldots \end{aligned} \tag{15}$$

Thus, by using (14) the tensor power series (6) can be represented by

$$\mathbf{G}(\mathbf{A}) = \varphi_0\mathbf{I} + \varphi_1\mathbf{A} + \varphi_2\mathbf{A}^2, \tag{16}$$

where the scalar coefficients φ_0, φ_1 and φ_2 result from

$$\varphi_r = \sum_{k=0}^{\infty} \chi_k c_k^{(r)}, \quad r = 0, 1, 2. \tag{17}$$

The relation (16) is well-known for isotropic functions of a symmetric tensor argument as the representation theorem (see e.g. Truesdell and Noll, 1965). It is essential that this representation is also valid for non-symmetric tensor functions defined in terms of the power series (6).

The direct differentiation of the tensor power series (6) with respect to the tensor argument yields by means of (10)

$$\mathbf{G}(\mathbf{A})_{,\mathbf{A}} = \sum_{n=1}^{\infty} \chi_n \sum_{k=0}^{n-1} \mathbf{A}^{n-1-k} \otimes \mathbf{A}^k. \tag{18}$$

Further, by virtue of (14) we obtain the representation

$$\mathbf{G}(\mathbf{A})_{,\mathbf{A}} = \sum_{r,t=0}^{2} \eta_{rt} \mathbf{A}^r \otimes \mathbf{A}^t, \tag{19}$$

where the scalar coefficients η_{rt} $(r, t = 0,\ 1,\ 2)$ can be calculated by

$$\eta_{rt} = \eta_{tr} = \sum_{n=1}^{\infty} \chi_n \sum_{k=0}^{n-1} c_{n-1-k}^{(r)} c_k^{(t)}, \quad r, t = 0, 1, 2. \tag{20}$$

Note, that for the exponential and other tensor functions defined by infinite power series, the use of the representations (16) and (19) requires the calculation of infinite coefficients series (17) and (20).

4. A closed-form solution

To avoid numerical calculation of the coefficient series (17) and (20) closed-form solutions for the tensor power series (6) and their derivative (18) can be obtained. To this end we again turn attention to the representations (16) and (19) and in particular to the coefficients φ_r and η_{rt} $(r, t = 0,\ 1,\ 2)$ appearing there. Under consideration of (15), (17) and (20) it is seen that these coefficients represent the functions of the principal invariants or eigenvalues of the tensor argument $\mathbf{A}$. Thus

$$\begin{aligned} \varphi_r &= \varphi_r(\mathrm{I}_\mathbf{A}, \mathrm{II}_\mathbf{A}, \mathrm{III}_\mathbf{A}) = \varphi_r(\lambda_1, \lambda_2, \lambda_3), \\ \eta_{rt} &= \eta_{rt}(\mathrm{I}_\mathbf{A}, \mathrm{II}_\mathbf{A}, \mathrm{III}_\mathbf{A}) = \eta_{rt}(\lambda_1, \lambda_2, \lambda_3), \quad r, t = 0, 1, 2. \end{aligned} \tag{21}$$

The crucial argument in the following consideration is that the functions (21) do not depend upon whether the tensor argument is symmetric or non-symmetric or whether it possesses a spectral decomposition (in diagonal form) or not. The coefficients (21) are uniquely determined in terms of the principal invariants or eigenvalues of the tensor argument. Hence, general expressions for the functions $\varphi_r(\lambda_1, \lambda_2, \lambda_3)$ and $\eta_{rt}(\lambda_1, \lambda_2, \lambda_3)$ $(r, t = 0,\ 1,\ 2)$ can be obtained considering the special case of a tensor argument with a spectral decomposition. These expressions will be then valid *for all tensors with the same eigenvalues* λ_1, λ_2 and λ_3!

In the case of distinct eigenvalues the procedure formulating the functions $\varphi_r(\lambda_1, \lambda_2, \lambda_3)$ $(r = 1, 2, 3)$ is rather standard one. It begins with the spectral decomposition of the tensor argument $\mathbf{A} \in \mathbf{Lin}$:

$$\mathbf{A} = \sum_{i=1}^{3} \lambda_i \mathbf{M}_i, \quad \mathbf{M}_k \mathbf{M}_l = \delta_{kl} \mathbf{M}_k, \qquad k, l = 1, 2, 3. \tag{22}$$

Note, that the tensor $\mathbf{A}$ is not generally symmetric such that two of its three eigenvalues λ_i and the corresponding eigenvalue-bases $\mathbf{M}_i$ $(i = 1, 2, 3)$ can be complex. In the case of distinct eigenvalues the eigenvalue-bases are uniquely determined by the Sylvester's formula:

$$\mathbf{M}_r = \prod_{\substack{s=1 \\ s \neq r}}^{3} \frac{\mathbf{A} - \lambda_s \mathbf{I}}{\lambda_r - \lambda_s}, \quad r = 1, 2, 3, \quad \lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1. \tag{23}$$

On the basis of the spectral decomposition (22) the tensor function $\mathbf{G}(\mathbf{A})$ (6) can also be given in the spectral form by

$$\mathbf{G}(\mathbf{A}) = \sum_{i=1}^{3} g(\lambda_i)\mathbf{M}_i, \tag{24}$$

where the complex function

$$g(\lambda) = \sum_{k=0}^{\infty} \chi_k \lambda^k \tag{25}$$

is usually referred to as diagonal function. Since the power series (25) converges on the spectrum of $\mathbf{A} \in \mathbf{D}\mathrm{lin}$ (see e.g. Gantmacher, 1959), the function $g(\lambda)$ defined by (25) is holomorphic and as a result of that infinitely often differentiable within the circle of convergence. Henceforth, we will also assume that the function $g(\lambda)$ can equivalently be given in a closed form without any reference to infinite power series (25).

Now, substituting (23) into (24) and comparing the result obtained with (16) yields the well-known relations (see e.g. Fitzgerald, 1980):

(i) *Distinct eigenvalues:* $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$,

$$\varphi_0 = \sum_{i=1}^{3} \frac{g(\lambda_i)\lambda_j\lambda_k}{D_i}, \quad \varphi_1 = -\sum_{i=1}^{3} \frac{g(\lambda_i)(\lambda_j + \lambda_k)}{D_i}, \quad \varphi_2 = \sum_{i=1}^{3} \frac{g(\lambda_i)}{D_i}, \tag{26}$$

where

$$D_i = (\lambda_i - \lambda_j)(\lambda_i - \lambda_k), \quad i \neq j \neq k \neq i = 1, 2, 3. \tag{27}$$

To specify the functions $\eta_{rt}(\lambda_1, \lambda_2, \lambda_3)$ $(r, t = 0,\ 1,\ 2)$ we first insert the spectral decomposition (22) into the relation (18). This leads to the closed-form solution for the derivative of non-symmetric tensor power series in terms of the eigenvalue-bases (see Itskov, 2002):

$$\mathbf{G}(\mathbf{A})_{,\mathbf{A}} = \sum_{i}^{3} g'(\lambda_i)\mathbf{M}_i \otimes \mathbf{M}_i + \sum_{i,j\neq i}^{3} \frac{g(\lambda_i) - g(\lambda_j)}{\lambda_i - \lambda_j}\mathbf{M}_i \otimes \mathbf{M}_j. \tag{28}$$

Considering in this solution the representation for the eigenvalue-bases (23) and comparing the result obtained with (19) delivers

(i) *Distinct eigenvalues:* $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$,

$$\eta_{00} = \sum_{i}^{3} \frac{\lambda_j^2\lambda_k^2 g'(\lambda_i)}{D_i^2} - \sum_{i,j\neq i}^{3} \frac{\lambda_i\lambda_j\lambda_k^2[g(\lambda_i) - g(\lambda_j)]}{(\lambda_i - \lambda_j)^3 D_k},$$

$$\eta_{01} = \eta_{10} = -\sum_{i}^{3} \frac{(\lambda_j + \lambda_k)\lambda_j\lambda_k g'(\lambda_i)}{D_i^2} + \sum_{i,j\neq i}^{3} \frac{(\lambda_j + \lambda_k)\lambda_i\lambda_k[g(\lambda_i) - g(\lambda_j)]}{(\lambda_i - \lambda_j)^3 D_k},$$

$$\eta_{02} = \eta_{20} = \sum_{i}^{3} \frac{\lambda_j\lambda_k g'(\lambda_i)}{D_i^2} - \sum_{i,j\neq i}^{3} \frac{\lambda_i\lambda_k[g(\lambda_i) - g(\lambda_j)]}{(\lambda_i - \lambda_j)^3 D_k},$$

$$\eta_{11} = \sum_{i}^{3} \frac{(\lambda_j + \lambda_k)^2 g'(\lambda_i)}{D_i^2} - \sum_{i,j\neq i}^{3} \frac{(\lambda_j + \lambda_k)(\lambda_i + \lambda_k)[g(\lambda_i) - g(\lambda_j)]}{(\lambda_i - \lambda_j)^3 D_k},$$

$$\eta_{12} = \eta_{21} = -\sum_{i}^{3} \frac{(\lambda_j + \lambda_k) g'(\lambda_i)}{D_i^2} + \sum_{i,j \neq i}^{3} \frac{(\lambda_i + \lambda_k)[g(\lambda_i) - g(\lambda_j)]}{(\lambda_i - \lambda_j)^3 D_k},$$

$$\eta_{22} = \sum_{i}^{3} \frac{g'(\lambda_i)}{D_i^2} - \sum_{i,j \neq i}^{3} \frac{g(\lambda_i) - g(\lambda_j)}{(\lambda_i - \lambda_j)^3 D_k}, \quad i \neq j \neq k \neq i. \tag{29}$$

In the case of distinct eigenvalues the expressions for the coefficients $\eta_{rt}(\lambda_1, \lambda_2, \lambda_3)$ $(r, t = 0, 1, 2)$ (29) coincide with that ones obtained by Carlson and Hoger (1986) for isotropic functions of symmetric tensor arguments.

The solution in the case of two repeated eigenvalues $\lambda_a \neq \lambda_b = \lambda_c$ results from (26), (27) and (29) as a limit at $\Delta = \lambda_b - \lambda_c \to 0$. Thus, we obtain

(ii) *Double coalescence of eigenvalues:* $\lambda_a \neq \lambda_b = \lambda_c = \lambda$, $(a \neq b \neq c)$,

$$\varphi_0 = \lambda \frac{\lambda g(\lambda_a) - \lambda_a g(\lambda)}{(\lambda_a - \lambda)^2} + \frac{\lambda_a g(\lambda)}{(\lambda_a - \lambda)} - \frac{\lambda \lambda_a g'(\lambda)}{(\lambda_a - \lambda)},$$

$$\varphi_1 = -2\lambda \frac{g(\lambda_a) - g(\lambda)}{(\lambda_a - \lambda)^2} + \frac{g'(\lambda)(\lambda_a + \lambda)}{(\lambda_a - \lambda)},$$

$$\varphi_2 = \frac{g(\lambda_a) - g(\lambda)}{(\lambda_a - \lambda)^2} - \frac{g'(\lambda)}{(\lambda_a - \lambda)}, \tag{30}$$

$$\eta_{00} = \frac{(2\lambda^2\lambda_a^2 - 6\lambda^3\lambda_a)[g(\lambda_a) - g(\lambda)]}{(\lambda_a - \lambda)^5} + \frac{\lambda^4 g'(\lambda_a) + (2\lambda^3\lambda_a + 4\lambda^2\lambda_a^2 - 4\lambda\lambda_a^3 + \lambda_a^4) g'(\lambda)}{(\lambda_a - \lambda)^4}$$
$$+ \frac{(2\lambda^2\lambda_a^2 - \lambda_a^3\lambda) g''(\lambda)}{(\lambda_a - \lambda)^3} + \frac{\lambda^2\lambda_a^2 g'''(\lambda)}{6(\lambda_a - \lambda)^2},$$

$$\eta_{01} = \eta_{10} = \frac{(3\lambda^3 + 7\lambda_a\lambda^2 - 2\lambda_a^2\lambda)[g(\lambda_a) - g(\lambda)]}{(\lambda_a - \lambda)^5} - \frac{2\lambda^3 g'(\lambda_a) + (\lambda^3 + 7\lambda_a\lambda^2 - 2\lambda_a^2\lambda) g'(\lambda)}{(\lambda_a - \lambda)^4}$$
$$- \frac{(4\lambda^2\lambda_a + \lambda_a^2\lambda - \lambda_a^3) g''(\lambda)}{2(\lambda_a - \lambda)^3} - \frac{\lambda_a\lambda(\lambda_a + \lambda) g'''(\lambda)}{6(\lambda_a - \lambda)^2},$$

$$\eta_{02} = \eta_{20} = \frac{(\lambda_a^2 - 3\lambda_a\lambda - 2\lambda^2)[g(\lambda_a) - g(\lambda)]}{(\lambda_a - \lambda)^5} + \frac{\lambda^2 g'(\lambda_a) + (\lambda^2 + 3\lambda_a\lambda - \lambda_a^2) g'(\lambda)}{(\lambda_a - \lambda)^4}$$
$$+ \frac{(3\lambda\lambda_a - \lambda_a^2) g''(\lambda)}{2(\lambda_a - \lambda)^3} + \frac{\lambda_a\lambda g'''(\lambda)}{6(\lambda_a - \lambda)^2},$$

$$\eta_{11} = -4\frac{\lambda(\lambda_a + 3\lambda)[g(\lambda_a) - g(\lambda)]}{(\lambda_a - \lambda)^5} + 4\frac{\lambda^2 g'(\lambda_a) + \lambda(\lambda_a + 2\lambda) g'(\lambda)}{(\lambda_a - \lambda)^4} + \frac{2\lambda(\lambda_a + \lambda) g''(\lambda)}{(\lambda_a - \lambda)^3} + \frac{(\lambda_a + \lambda)^2 g'''(\lambda)}{6(\lambda_a - \lambda)^2},$$

$$\eta_{12} = \eta_{21} = \frac{(\lambda_a + 7\lambda)[g(\lambda_a) - g(\lambda)]}{(\lambda_a - \lambda)^5} - \frac{2\lambda g'(\lambda_a) + (\lambda_a + 5\lambda) g'(\lambda)}{(\lambda_a - \lambda)^4} - \frac{(\lambda_a + 3\lambda) g''(\lambda)}{2(\lambda_a - \lambda)^3} - \frac{(\lambda_a + \lambda) g'''(\lambda)}{6(\lambda_a - \lambda)^2},$$

$$\eta_{22} = -4\frac{g(\lambda_a) - g(\lambda)}{(\lambda_a - \lambda)^5} + \frac{g'(\lambda_a) + 3g'(\lambda)}{(\lambda_a - \lambda)^4} + \frac{g''(\lambda)}{(\lambda_a - \lambda)^3} + \frac{g'''(\lambda)}{6(\lambda_a - \lambda)^2}. \tag{31}$$

Similarly we proceed in the case of three repeated eigenvalues. To this end we consider the limit $\Delta = \lambda_a - \lambda \to 0$ in (30) and (31). This yields

(iii) *Triple coalescence of eigenvalues:* $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$,

$$\varphi_0 = g(\lambda) - \lambda g'(\lambda) + \frac{1}{2}\lambda^2 g''(\lambda), \quad \varphi_1 = g'(\lambda) - \lambda g''(\lambda), \quad \varphi_2 = \frac{1}{2}g''(\lambda), \tag{32}$$

$$\eta_{00} = g'(\lambda) - \lambda g''(\lambda) + \frac{\lambda^2 g'''(\lambda)}{2} - \frac{\lambda^3 g^{\mathrm{IV}}(\lambda)}{12} + \frac{\lambda^4 g^{\mathrm{V}}(\lambda)}{120},$$

$$\eta_{01} = \eta_{10} = \frac{g''(\lambda)}{2} - \frac{\lambda g'''(\lambda)}{2} + \frac{\lambda^2 g^{\mathrm{IV}}(\lambda)}{8} - \frac{\lambda^3 g^{\mathrm{V}}(\lambda)}{60},$$

$$\eta_{02} = \eta_{20} = \frac{g'''(\lambda)}{6} - \frac{\lambda g^{\mathrm{IV}}(\lambda)}{24} + \frac{\lambda^2 g^{\mathrm{V}}(\lambda)}{120}, \quad \eta_{11} = \frac{g'''(\lambda)}{6} - \frac{\lambda g^{\mathrm{IV}}(\lambda)}{6} + \frac{\lambda^2 g^{\mathrm{V}}(\lambda)}{30},$$

$$\eta_{12} = \eta_{21} = \frac{g^{\mathrm{IV}}(\lambda)}{24} - \frac{\lambda g^{\mathrm{V}}(\lambda)}{60}, \quad \eta_{22} = \frac{g^{\mathrm{V}}(\lambda)}{120}. \tag{33}$$

In the following discussion the issue of continuity of the tensor function $\mathbf{G}(\mathbf{A})$ given by (16), (26), (27), (30) and (32) is of major importance. For isotropic functions of symmetric tensors the problem has been addressed by Man (1994, 1995). Here, it should namely be shown that the solutions (30) and (32) obtained for the cases of repeated eigenvalues do not depend on the direction of the limits

$$\text{(ii)} \quad (\lambda_a, \lambda_b, \lambda_c) \to (\lambda_a, \lambda, \lambda), \quad \text{(iii)} \quad (\lambda_a, \lambda_b, \lambda_c) \to (\lambda, \lambda, \lambda). \tag{34}$$

It can be seen that the complex functions φ_r ($r = 0$, 1, 2) defined by (26) and (27) are holomorphic at least for distinct eigenvalues $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$. The cases of repeated eigenvalues should be treated separately. First, we consider the case of double coalescence of eigenvalues (ii). Let $\lambda_b = \lambda$, $\Delta = \lambda_c - \lambda$ and $\varphi_r(\Delta) = \varphi_r(\lambda_a, \lambda, \lambda + \Delta)$ ($r = 0, 1, 2$). Keeping in mind that the diagonal function $g(\lambda)$ is holomorphic one can expand the functions $\varphi_r(\Delta)$ in the Taylor power series

$$\varphi_r(\Delta) = \sum_{n=0}^{\infty} a_n^{(r)} \Delta^n, \quad r = 0, 1, 2 \tag{35}$$

in the vicinity of the point $\Delta = 0$. Note that in the corresponding Laurent series the remaining terms with negative powers identically vanish. Hence, we infer that the functions $\varphi_r(\Delta)$ are holomorphic including the point $\Delta = 0$. Thus, the solution in the case of two repeated eigenvalues $\varphi_r(0) = a_0^{(r)}$ ($r = 0$, 1, 2) does not depend on the direction of the limit $(34)_1$ and is expressed by (30).

Similarly we proceed in the case of three repeated eigenvalues (iii). Let $\lambda_a = \lambda$, $\Delta_1 = \lambda_b - \lambda$, $\Delta_2 = \lambda_c - \lambda$ and $\varphi_r(\Delta_1, \Delta_2) = \varphi_r(\lambda, \lambda + \Delta_1, \lambda + \Delta_2)$ ($r = 0, 1, 2$). In the vicinity of the point $(0, 0)$ the complex functions $\varphi_r(\Delta_1, \Delta_2)$ ($r = 0$, 1, 2) are expandable in the Taylor power series

$$\varphi_r(\Delta_1, \Delta_2) = \sum_{k,n=0}^{\infty} a_{kn}^{(r)} \Delta_1^k \Delta_2^n, \quad r = 0, 1, 2 \tag{36}$$

and on account of this are holomorphic in this point as well. For example, for the function φ_2 the series (36) begins as follows

$$\varphi_2(\Delta_1, \Delta_2) = \frac{1}{2}g''(\lambda) + \frac{1}{6}g'''(\lambda)(\Delta_1 + \Delta_2) + \frac{1}{24}g^{\mathrm{IV}}(\lambda)(\Delta_1^2 + \Delta_1\Delta_2 + \Delta_2^2) + \cdots \tag{37}$$

We observe that the solution in the case of three repeated eigenvalues $\varphi_r(0,0) = a_{00}^{(r)}$ $(r = 0,1,2)$ expressed by (32) is also independent of the direction of the limit $(34)_2$. Thus, the tensor function $\mathbf{G}(\mathbf{A})$ given by (16), (26), (27), (30) and (32) is continuous on the whole definition domain **D**lin. Using the same reasoning we infer that this function is also continuously differentiable since the solution for $\mathbf{G}(\mathbf{A})_{,\mathbf{A}}$ (19), (29), (31) and (33) is continuous on **D**lin as well.

Remark 4.1. Instead of the power series (6) an isotropic tensor function can alternatively be defined by the representation (16) where the coefficients φ_r $(r = 0, 1, 2)$ are expressed by (26), (27), (30) and (32) and the diagonal function $g(\lambda)$ is explicitly given in a closed form (without infinite series). Such a definition can be of advantage if the corresponding infinite tensor power series of the form (6) converges only on a narrow subset of **L**in. This can be illustrated e.g. by the logarithmic tensor function. Indeed, the tensor power series

$$\ln(\mathbf{A} + \mathbf{I}) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\mathbf{A}^n}{n} \tag{38}$$

converges if $|\lambda_i| < 1$ $(i = 1, 2, 3)$ which vastly restricts the definition domain of the logarithmic tensor function (38). A more preferable definition is due to the complex logarithmic diagonal function (in the sense of the principal value)

$$g(\lambda) = \ln \lambda, \quad \lambda \neq 0 \tag{39}$$

in the representation (16), (26), (27), (30) and (32). The logarithmic tensor function obtained in this manner is defined for all invertible second-order tensors $(\forall \mathbf{A} \in \mathbf{L}\mathrm{in} : \det \mathbf{A} \neq 0)$.

In the case of distinct eigenvalues the tensor argument $\mathbf{A} \in \mathbf{L}\mathrm{in}$ always possesses a spectral decomposition. Thereby, the derivative of an isotropic tensor function defined by power series (6) can alternatively be obtained by means of the closed formula in terms of the eigenvalue-bases (eigenprojections) even if the tensor **A** is non-symmetric (see Itskov, 2002). On the contrary, if some eigenvalues of the non-symmetric tensor argument are multiple it can possess no spectral decomposition. This is namely the case if a repeated eigenvalue is defective such that its algebraic multiplicity exceeds the geometric multiplicity i.e. the number of linearly independent eigenvectors associated with this eigenvalue (see e.g. Golub and Van Loan, 1996). In this case, the solution (19), (31) and (33) represents, to our best knowledge, the only possibility to calculate the derivative of infinite tensor power series in a closed form.

5. An alternative form of the closed-form representation for the derivative of the isotropic tensor function

The derivative of the isotropic tensor function $\mathbf{G}(\mathbf{A})$ can alternatively be obtained by directly differentiating the representation (16), (26) and (27) with respect to the tensor argument $\mathbf{A} \in \mathbf{L}\mathrm{in}$. The advantage of this procedure is that the function $\mathbf{G}(\mathbf{A})$ can be given by (16), (26), (27), (30) and (32) without any reference to the tensor power series (6) which can extend its definition domain (see Remark 4.1).

The critical issue of this procedure is the differentiability of eigenvalues. If distinct, the eigenvalues of a second-order tensor are proved to be differentiable (for the proof see e.g. Lax, 1997). The derivatives of eigenvalues can be expressed using the Vieta's theorem

$$\lambda_1 + \lambda_2 + \lambda_3 = \mathrm{I}_{\mathbf{A}}, \quad \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = \mathrm{II}_{\mathbf{A}}, \quad \lambda_1\lambda_2\lambda_3 = \mathrm{III}_{\mathbf{A}}. \tag{40}$$

The differentiation of these relations with respect to $\mathbf{A}$ yields the linear equation system

$$\begin{bmatrix} 1 & 1 & 1 \\ \lambda_2+\lambda_3 & \lambda_3+\lambda_1 & \lambda_1+\lambda_2 \\ \lambda_2\lambda_3 & \lambda_3\lambda_1 & \lambda_1\lambda_2 \end{bmatrix} \begin{Bmatrix} \lambda_1,_{\mathbf{A}} \\ \lambda_2,_{\mathbf{A}} \\ \lambda_3,_{\mathbf{A}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{I} \\ \mathrm{I}_{\mathbf{A}}\mathbf{I}-\mathbf{A}^{\mathrm{T}} \\ \left[\mathbf{A}^2-\mathrm{I}_{\mathbf{A}}\mathbf{A}+\mathrm{II}_{\mathbf{A}}\mathbf{I}\right]^{\mathrm{T}} \end{Bmatrix}, \tag{41}$$

which has in the case of distinct eigenvalues the following unique solution

$$\lambda_i,_{\mathbf{A}} = \frac{1}{D_i}(\mathbf{A}^{\mathrm{T}}-\lambda_j\mathbf{I})(\mathbf{A}^{\mathrm{T}}-\lambda_k\mathbf{I}) = \mathbf{M}_i^{\mathrm{T}}, \quad i\neq j\neq k\neq i=1,2,3. \tag{42}$$

Remark 5.1. The identity $\lambda_i,_{\mathbf{A}} = \mathbf{M}_i^{\mathrm{T}}$ is well-known in the perturbation theory for linear operators (see e.g. Kato, 1966) but we included it to make the exposition self-contained.

Using the relation (42) and with aid of the product rule $(11)_2$ we obtain another representation

(i) *Distinct eigenvalues:* $\lambda_1\neq\lambda_2\neq\lambda_3\neq\lambda_1$,

$$\mathbf{G}(\mathbf{A}),_{\mathbf{A}} = \sum_{i=1}^{3}\alpha_i[(\mathbf{A}-\lambda_j\mathbf{I})(\mathbf{A}-\lambda_k\mathbf{I})]\times[(\mathbf{A}-\lambda_j\mathbf{I})(\mathbf{A}-\lambda_k\mathbf{I})]^{\mathrm{T}} + \varphi_1\mathscr{I} + \varphi_2(\mathbf{A}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{A}),$$
$$i\neq j\neq k\neq i, \tag{43}$$

where φ_1 and φ_2 are given by (26) and

$$\alpha_i = \frac{1}{D_i^2}\left(g'(\lambda_i)-\frac{g(\lambda_i)}{\lambda_i-\lambda_j}-\frac{g(\lambda_i)}{\lambda_i-\lambda_k}\right)+\frac{g(\lambda_j)}{D_k(\lambda_i-\lambda_j)^3}+\frac{g(\lambda_k)}{D_j(\lambda_i-\lambda_k)^3}, \quad i\neq j\neq k\neq i=1,2,3. \tag{44}$$

It is seen that the representation (16) is differentiable on the definition domain $\mathbf{D}\mathrm{lin}\subset\mathbf{L}\mathrm{in}$ of the tensor function $\mathbf{G}(\mathbf{A})$ at least for the tensor arguments with distinct eigenvalues. In the case of repeated eigenvalues the differentiability of (16) can be shown by means of the Ball's lemma (Ball, 1984). Accordingly, a tensor function $\mathbf{G}(\mathbf{A})$ is differentiable on a closed sparse subset $\mathbf{S}\mathrm{lin}$ of the open definition domain $\mathbf{D}\mathrm{lin}$, if this function is continuous on $\mathbf{D}\mathrm{lin}$ and continuously differentiable on the complement $\overline{\mathbf{S}\mathrm{lin}} = \{\mathbf{A}\in\mathbf{D}\mathrm{lin} : \mathbf{A}\notin\mathbf{S}\mathrm{lin}\subset\mathbf{D}\mathrm{lin}\}$ and if there exist the limit

$$\lim_{\mathbf{A}\to\mathbf{B}}\mathbf{G}(\mathbf{A}),_{\mathbf{A}}, \quad \forall\mathbf{B}\in\mathbf{S}\mathrm{lin}\subset\mathbf{D}\mathrm{lin}, \ \ \mathbf{A}\in\overline{\mathbf{S}\mathrm{lin}}. \tag{45}$$

In the previous section we have shown that the function $\mathbf{G}(\mathbf{A})$ defined by (16), (26), (27), (30) and (32) is continuous on its definition domain. Further, let $\mathbf{S}\mathrm{lin}\subset\mathbf{D}\mathrm{lin}$ be a subset of tensors with repeated eigenvalues. We first prove that $\mathbf{S}\mathrm{lin}$ is closed and sparse. Ball (1984) has shown that a sufficient condition for such a set to be a closed and sparse is that it can be defined by means of a non-constant polynomial $p(\mathbf{A})$ as $\mathbf{S}\mathrm{lin} = \{\mathbf{A}\in\mathbf{D}\mathrm{lin} : p(\mathbf{A})=0\}$. The conditions of at least two repeated eigenvalues $\lambda_a\neq\lambda_b=\lambda_c=\lambda$ can be formulated in view of (40) as follows

$$\lambda_a+2\lambda=\mathrm{I}_{\mathbf{A}}, \quad \lambda^2+2\lambda_a\lambda=\mathrm{II}_{\mathbf{A}}, \quad \lambda_a\lambda^2=\mathrm{III}_{\mathbf{A}}. \tag{46}$$

Eliminating the eigenvalues we obtain the only condition

$$\pm(\mathrm{I}_{\mathbf{A}}^2-3\mathrm{II}_{\mathbf{A}})^{3/2} = \frac{1}{2}(27\mathrm{III}_{\mathbf{A}}-9\mathrm{II}_{\mathbf{A}}\mathrm{I}_{\mathbf{A}}+2\mathrm{I}_{\mathbf{A}}^3). \tag{47}$$

Hence, the subset $\mathbf{S}\mathrm{lin}$ can be formed by the zero set of the scalar-valued function

$$p(\mathbf{A}) = (\mathrm{I}_{\mathbf{A}}^2-3\mathrm{II}_{\mathbf{A}})^3-\frac{1}{4}(27\mathrm{III}_{\mathbf{A}}-9\mathrm{II}_{\mathbf{A}}\mathrm{I}_{\mathbf{A}}+2\mathrm{I}_{\mathbf{A}}^3)^2. \tag{48}$$

Accordingly, $p(\mathbf{A})$ is the polynomial function of the principal invariants and thus is polynomial with respect to $\mathbf{A}$.

In analogy a polynomial function can also be constructed for the subset **Slin** characterized by the triple coalescence of eigenvalues. In this case we obtain instead of (48)

$$p(\mathbf{A}) = (\mathrm{I}_{\mathbf{A}}^2 - 3\mathrm{II}_{\mathbf{A}})^2 + (27\mathrm{III}_{\mathbf{A}} - \mathrm{I}_{\mathbf{A}}^3)^2. \tag{49}$$

It remains to show that the derivative (43) and (44) has a limit at $p(\mathbf{A}) \to 0$. First we rewrite (43) as follows

$$\begin{aligned}\mathbf{G}(\mathbf{A})_{,\mathbf{A}} &= \mathbf{A}^2 \times \mathbf{A}^{\mathrm{T}^2} \sum_{i=1}^{3} \alpha_i - (\mathbf{A}^2 \times \mathbf{A}^{\mathrm{T}} + \mathbf{A} \times \mathbf{A}^{\mathrm{T}^2}) \sum_{i=1}^{3} \alpha_i(\lambda_j + \lambda_k) + (\mathbf{A}^2 \times \mathbf{I} + \mathbf{I} \times \mathbf{A}^{\mathrm{T}^2}) \sum_{i=1}^{3} \alpha_i \lambda_j \lambda_k \\ &\quad + \mathbf{A} \times \mathbf{A}^{\mathrm{T}} \sum_{i=1}^{3} \alpha_i(\lambda_j + \lambda_k)^2 - (\mathbf{A} \times \mathbf{I} + \mathbf{I} \times \mathbf{A}^{\mathrm{T}}) \sum_{i=1}^{3} \alpha_i \lambda_j \lambda_k (\lambda_j + \lambda_k) + \mathbf{I} \times \mathbf{I} \sum_{i=1}^{3} \alpha_i \lambda_j^2 \lambda_k^2 \\ &\quad + \varphi_1 \mathscr{I} + \varphi_2(\mathbf{A} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{A}), \quad i \neq j \neq k \neq i.\end{aligned} \tag{50}$$

Using the procedure described in the previous section it can be shown that the scalar coefficients appearing in (50) represent holomorphic functions of eigenvalues even if the eigenvalues coincide. Thus, considering the limit case $\Delta = \lambda_b - \lambda_c \to 0$ in (50) we may write

(ii) *Double coalescence of eigenvalues*: $\lambda_a \neq \lambda_b = \lambda_c = \lambda$, $(a \neq b \neq c)$,

$$\begin{aligned}\mathbf{G}(\mathbf{A})_{,\mathbf{A}} &= \left(\Gamma - \frac{\Phi}{(\lambda_a - \lambda)^2}\right)(\mathbf{A} - \lambda\mathbf{I})^2 \times (\mathbf{A}^{\mathrm{T}} - \lambda\mathbf{I})^2 + \Upsilon[(\mathbf{A} - \lambda\mathbf{I})(\mathbf{A} - \lambda_a\mathbf{I})] \times [(\mathbf{A} - \lambda\mathbf{I})(\mathbf{A} - \lambda_a\mathbf{I})]^{\mathrm{T}} \\ &\quad + \Phi[\mathbf{I} \times (\mathbf{A}^{\mathrm{T}^2} - \mathrm{I}_{\mathbf{A}}\mathbf{A}^{\mathrm{T}} + \mathrm{II}_{\mathbf{A}}\mathbf{I}) - \mathbf{A} \times (\mathrm{I}_{\mathbf{A}}\mathbf{I} - \mathbf{A}^{\mathrm{T}}) + \mathbf{A}^2 \times \mathbf{I}] + \varphi_1 \mathscr{I} + \varphi_2(\mathbf{A} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{A}),\end{aligned} \tag{51}$$

where φ_1 and φ_2 are defined according to (30) and

$$\begin{aligned}\Gamma &= \frac{1}{(\lambda_a - \lambda)^4}\left(g'(\lambda_a) + g'(\lambda) - 2\frac{g(\lambda_a) - g(\lambda)}{\lambda_a - \lambda}\right), \quad \Upsilon = \frac{1}{(\lambda_a - \lambda)^2}\left(\frac{1}{6}g'''(\lambda) - \Phi\right), \\ \Phi &= \frac{1}{\lambda_a - \lambda}\left(-\frac{1}{2}g''(\lambda) - \frac{g'(\lambda)}{\lambda_a - \lambda} + \frac{g(\lambda_a) - g(\lambda)}{(\lambda_a - \lambda)^2}\right).\end{aligned} \tag{52}$$

Using a similar procedure we further obtain by setting in (51) and (52) $\Delta = \lambda_a - \lambda \to 0$

(iii) *Triple coalescence of eigenvalues:* $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$,

$$\begin{aligned}\mathbf{G}(\mathbf{A})_{,\mathbf{A}} &= \frac{g'''(\lambda)}{6}[\mathbf{I} \times (\mathbf{A}^2 - 3\lambda\mathbf{A} + 3\lambda^2\mathbf{I})^{\mathrm{T}} - \mathbf{A} \times (3\lambda\mathbf{I} - \mathbf{A})^{\mathrm{T}} + \mathbf{A}^2 \times \mathbf{I}] + \frac{g^{\mathrm{IV}}(\lambda)}{24}\Big[(\mathbf{A} - \lambda\mathbf{I}) \times (\mathbf{A}^{\mathrm{T}} - \lambda\mathbf{I})^2 \\ &\quad + (\mathbf{A} - \lambda\mathbf{I})^2 \times (\mathbf{A}^{\mathrm{T}} - \lambda\mathbf{I})\Big] + \frac{g^{\mathrm{V}}(\lambda)}{120}(\mathbf{A} - \lambda\mathbf{I})^2 \times \left(\mathbf{A}^{\mathrm{T}} - \lambda\mathbf{I}\right)^2 + [g'(\lambda) - \lambda g''(\lambda)]\mathscr{I} \\ &\quad + \frac{1}{2}g''(\lambda)(\mathbf{A} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{A}).\end{aligned} \tag{53}$$

Thus, we have obtained the representation for the derivative of the tensor function $\mathbf{G}(\mathbf{A})$ in two different forms (19), (29), (31) and (33) on the one hand and (43), (44) and (51)–(53) on the other hand. It can be shown that they are equivalent. To this end we first prove some universal tensor identities connecting fourth-order tensors constructed by the tensor products "$\times$" and "$\otimes$" (4). The first identity directly results from the differentiation of the Cayley–Hamilton relation (12). Thus, we obtain by virtue of (10) and (11)

$$\begin{aligned}&\mathbf{A}^2 \otimes \mathbf{I} + \mathbf{A} \otimes \mathbf{A} + \mathbf{I} \otimes \mathbf{A}^2 - \mathrm{I}_{\mathbf{A}}(\mathbf{A} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{A}) + \mathrm{II}_{\mathbf{A}}\mathscr{I} \\ &\quad - \mathbf{A}^2 \times \mathbf{I} + \mathbf{A} \times (\mathrm{I}_{\mathbf{A}}\mathbf{I} - \mathbf{A}^{\mathrm{T}}) - \mathbf{I} \times (\mathbf{A}^{\mathrm{T}^2} - \mathrm{I}_{\mathbf{A}}\mathbf{A}^{\mathrm{T}} + \mathrm{II}_{\mathbf{A}}\mathbf{I}) = \mathscr{O} \quad \forall \mathbf{A} \in \mathbf{Lin},\end{aligned} \tag{54}$$

where $\mathscr{O}$ denotes the fourth-order zero tensor characterized by $\mathscr{O} : \mathbf{A} = \mathbf{0}\ \forall \mathbf{A} \in \mathbf{Lin}$. The relation (54) can be considered as another form of the Rivlin's identity (Rivlin, 1955) written in terms of fourth-order tensors. Indeed, through the double contraction of (54) with an arbitrary second-order tensor $\mathbf{B}$ we obtain under consideration of (4)

$$\begin{aligned}&\mathbf{A}^2\mathbf{B} + \mathbf{ABA} + \mathbf{BA}^2 - \mathrm{I}_\mathbf{A}(\mathbf{AB} + \mathbf{BA}) + \mathrm{II}_\mathbf{A}\mathbf{B} - \mathbf{A}^2\mathrm{I}_\mathbf{B} + \mathbf{A}[\mathrm{I}_\mathbf{A}\mathrm{I}_\mathbf{B} - \mathrm{tr}(\mathbf{AB})]\\ &\quad - \mathbf{I}\left[\mathrm{tr}\left(\mathbf{A}^2\mathbf{B}\right) - \mathrm{I}_\mathbf{A}\mathrm{tr}(\mathbf{AB}) + \mathrm{II}_\mathbf{A}\mathrm{tr}\,\mathbf{B}\right] = \mathbf{0} \quad \forall \mathbf{A}, \mathbf{B} \in \mathbf{Lin}.\end{aligned} \tag{55}$$

The second important identity reads as

$$\begin{aligned}&\left[(\mathbf{A} - \lambda_i\mathbf{I})\left(\mathbf{A} - \lambda_j\mathbf{I}\right)\right] \times \left[(\mathbf{A} - \lambda_i\mathbf{I})\left(\mathbf{A} - \lambda_j\mathbf{I}\right)\right]^\mathrm{T} = \left[(\mathbf{A} - \lambda_i\mathbf{I})\left(\mathbf{A} - \lambda_j\mathbf{I}\right)\right] \otimes \left[(\mathbf{A} - \lambda_i\mathbf{I})\left(\mathbf{A} - \lambda_j\mathbf{I}\right)\right],\\ &\quad i \neq j = 1, 2, 3,\end{aligned} \tag{56}$$

and follows from the relation (cf. Carlson and Hoger, 1986)

$$\mathbf{M}_i \times \mathbf{M}_i^\mathrm{T} = \mathbf{M}_i \otimes \mathbf{M}_i \quad (\text{no sum. over } i = 1, 2, 3). \tag{57}$$

Indeed, expressing the eigenvalue-bases $\mathbf{M}_i$ of the tensor $\mathbf{A} \in \mathbf{Lin}$ through its right $\boldsymbol{n}_i$ and left $\boldsymbol{m}_i$ ($i = 1, 2, 3$) eigenvectors

$$\mathbf{M}_i = \boldsymbol{n}_i \otimes \boldsymbol{m}_i \quad (\text{no sum. over } i = 1, 2, 3), \tag{58}$$

and contracting the right and left hand side of (57) with an arbitrary second-order tensor $\mathbf{X} \in \mathbf{Clin}$ we obtain the same expressions

$$\begin{aligned}&\left(\mathbf{M}_i \times \mathbf{M}_i^\mathrm{T}\right) : \mathbf{X} = \mathrm{tr}(\mathbf{M}_i\mathbf{X})\mathbf{M}_i = (\boldsymbol{m}_i\mathbf{X}\boldsymbol{n}_i)\boldsymbol{n}_i \otimes \boldsymbol{m}_i,\\ &(\mathbf{M}_i \otimes \mathbf{M}_i) : \mathbf{X} = \mathbf{M}_i\mathbf{X}\mathbf{M}_i = (\boldsymbol{m}_i\mathbf{X}\boldsymbol{n}_i)\boldsymbol{n}_i \otimes \boldsymbol{m}_i \quad \forall \mathbf{X} \in \mathbf{Clin} \quad (\text{no sum. over } i = 1, 2, 3).\end{aligned} \tag{59}$$

Further, inserting the representation for the eigenvalue-bases (23) into (57) yields

$$\prod_{\substack{s=1\\ s\neq r}}^{3} \frac{\mathbf{A} - \lambda_s\mathbf{I}}{\lambda_r - \lambda_s} \times \prod_{\substack{s=1\\ s\neq r}}^{3} \frac{\mathbf{A}^\mathrm{T} - \lambda_s\mathbf{I}}{\lambda_r - \lambda_s} = \prod_{\substack{s=1\\ s\neq r}}^{3} \frac{\mathbf{A} - \lambda_s\mathbf{I}}{\lambda_r - \lambda_s} \otimes \prod_{\substack{s=1\\ s\neq r}}^{3} \frac{\mathbf{A} - \lambda_s\mathbf{I}}{\lambda_r - \lambda_s}, \quad r = 1, 2, 3. \tag{60}$$

Thus, it is observable that the identity (56) holds at least for distinct eigenvalues of the tensor $\mathbf{A}$. Considering the cases of repeated eigenvalues as a limit at $\lambda_1 - \lambda_2$ and (or) $\lambda_2 - \lambda_3$ tending to zero and keeping in mind that the nominators in (60) can be represented as continuous functions of the eigenvalues we infer that the identity (56) is generally valid.

Finally, considering the identities (54) and (56) in the solution (43), (44) and (51)–(53) one immediately arrives at the representation (19), (29), (31) and (33).

6. Special case of the tensor argument possessing a spectral decomposition

The representations for the tensor power series and their derivative (16), (19), (26), (27), (29)–(33), (43), (44) and (51)–(53) are valid for all second-order tensors even for those ones that have defective eigenvalues and thus possess no spectral decomposition. To verify our results we specify the representations (30)–(33) and (51)–(53) for the special case of the tensor argument possessing a spectral decomposition wherein solutions for symmetric isotropic tensor functions can be used for the comparison.

In this special case the representations (30)–(33) and (51)–(53) can be simplified by means of the identities $\mathbf{A}^2 = (\lambda_a + \lambda)\mathbf{A} - \lambda_a\lambda\mathbf{I}$ for the case of double coalescence of eigenvalues ($\lambda_a \neq \lambda_b = \lambda_c = \lambda$) and $\mathbf{A} = \lambda\mathbf{I}$, $\mathbf{A}^2 = \lambda^2\mathbf{I}$ for the case of triple coalescence of eigenvalues ($\lambda_1 = \lambda_2 = \lambda_3 = \lambda$). Thus, we obtain the well-known representation for symmetric isotropic tensor functions (see e.g. Carlson and Hoger, 1986):

(ii) *Double coalescence of eigenvalues:* $\lambda_a \neq \lambda_b = \lambda_c = \lambda$, $\mathbf{A} = \lambda_a \mathbf{M}_a + \lambda(\mathbf{I} - \mathbf{M}_a)$,

$$\mathbf{G}(\mathbf{A}) = \frac{\lambda_a g(\lambda) - \lambda g(\lambda_a)}{\lambda_a - \lambda}\mathbf{I} + \frac{g(\lambda_a) - g(\lambda)}{\lambda_a - \lambda}\mathbf{A}, \tag{61}$$

$$\begin{aligned}\mathbf{G}(\mathbf{A})_{,\mathbf{A}} = {} & \left[-2\lambda_a\lambda\frac{g(\lambda_a) - g(\lambda)}{(\lambda_a - \lambda)^3} + \frac{\lambda^2 g'(\lambda_a) + \lambda_a^2 g'(\lambda)}{(\lambda_a - \lambda)^2}\right]\mathbf{I}\otimes\mathbf{I} + \left[(\lambda_a + \lambda)\frac{g(\lambda_a) - g(\lambda)}{(\lambda_a - \lambda)^3}\right.\\ & \left. - \frac{\lambda g'(\lambda_a) + \lambda_a g'(\lambda)}{(\lambda_a - \lambda)^2}\right](\mathbf{A}\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{A}) + \left[-2\frac{g(\lambda_a) - g(\lambda)}{(\lambda_a - \lambda)^3} + \frac{g'(\lambda_a) + g'(\lambda)}{(\lambda_a - \lambda)^2}\right]\mathbf{A}\otimes\mathbf{A},\end{aligned} \tag{62}$$

or

$$\mathbf{G}(\mathbf{A})_{,\mathbf{A}} = \Gamma(\lambda_a - \lambda)^2(\mathbf{A} - \lambda\mathbf{I}) \times (\mathbf{A}^{\mathrm{T}} - \lambda\mathbf{I}) + \varphi_1\mathscr{I} + \varphi_2(\mathbf{A}\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{A}), \tag{63}$$

where φ_1, φ_2 and Γ are given by (30) and (52), respectively.

(iii) *Triple coalescence of eigenvalues:* $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, $\mathbf{A} = \lambda\mathbf{I}$,

$$\mathbf{G}(\mathbf{A}) = g(\lambda)\mathbf{I}, \quad \mathbf{G}(\mathbf{A})_{,\mathbf{A}} = g'(\lambda)\mathscr{I}. \tag{64}$$

7. Example

To illustrate the application of the closed-form solutions (16), (19), (26), (27), (29)–(33), (43), (44) and (51)–(53) we consider the exponential function of the velocity gradient under simple shear. In this loading case the deformation gradient can be given with respect to the Cartesian co-ordinate system by

$$\mathbf{F} = \mathrm{F}^{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j, \quad \mathrm{F}^{ij} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}, \tag{65}$$

where γ denotes the shear number. Thus, the velocity gradient $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ takes the form

$$\mathbf{L} = \mathrm{L}^{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j, \quad \mathrm{L}^{ij} = \begin{bmatrix} 0 & \dot{\gamma} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{66}$$

It observable that $\mathbf{L}$ has the triple eigenvalue

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda = 0, \tag{67}$$

which is defective, since it is associated with only two linearly independent (right) eigenvectors expressible in the normalized form by

$$\boldsymbol{n}_1 = \boldsymbol{e}_1, \quad \boldsymbol{n}_2 = \boldsymbol{e}_3. \tag{68}$$

Hence, the tensor $\mathbf{L}$ (66) possesses no spectral decomposition (in diagonal form) such that its isotropic functions as well as their derivative cannot be obtained by means of representations derived with the aid of the eigenprojections. Instead, we exploit the closed-form solution presented above and compare the result with that one due to the direct calculation of the tensor power series (8) and (18).

First, according to (16) and (32) we obtain for the case of three repeated eigenvalues

$$\exp(\mathbf{L}) = \mathrm{e}^{\lambda}\left(\frac{1}{2}\lambda^2 - \lambda + 1\right)\mathbf{I} + \mathrm{e}^{\lambda}(1 - \lambda)\mathbf{L} + \frac{1}{2}\mathrm{e}^{\lambda}\mathbf{L}^2. \tag{69}$$

Thus, under consideration of (67)

$$\exp(\mathbf{L}) = \mathbf{I} + \mathbf{L} + \frac{1}{2}\mathbf{L}^2. \tag{70}$$

The same result can also be obtained directly from the definition of the exponential function (8) by exploiting the relation

$$\mathbf{L}^n = \mathbf{0}, \quad n = 2, 3, \ldots, \tag{71}$$

following from the structure (nilpotent) of the tensor $\mathbf{L}$ (66). The derivative of the exponential function results from (19) and (33)

$$\begin{aligned}\exp(\mathbf{L}),_{\mathbf{L}} = \mathrm{e}^{\lambda}\Bigg[&\left(1 - \lambda + \frac{\lambda^2}{2} - \frac{\lambda^3}{12} + \frac{\lambda^4}{120}\right)\mathscr{I} + \left(\frac{1}{2} - \frac{\lambda}{2} + \frac{\lambda^2}{8} - \frac{\lambda^3}{60}\right)(\mathbf{L}\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{L}) \\ &+ \left(\frac{1}{6} - \frac{\lambda}{6} + \frac{\lambda^2}{30}\right)\mathbf{L}\otimes\mathbf{L} + \left(\frac{1}{6} - \frac{\lambda}{24} + \frac{\lambda^2}{120}\right)\left(\mathbf{L}^2\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{L}^2\right) \\ &+ \left(\frac{1}{24} - \frac{\lambda}{60}\right)(\mathbf{L}^2\otimes\mathbf{L} + \mathbf{L}\otimes\mathbf{L}^2) + \frac{1}{120}\mathbf{L}^2\otimes\mathbf{L}^2\Bigg]\end{aligned} \tag{72}$$

or from (53) in another form

$$\begin{aligned}\exp(\mathbf{L}),_{\mathbf{L}} = &\frac{\mathrm{e}^{\lambda}}{6}\left[\mathbf{I}\times\left(\mathbf{L}^2 - 3\lambda\mathbf{L} + 3\lambda^2\mathbf{I}\right)^{\mathrm{T}} - \mathbf{L}\times(3\lambda\mathbf{I} - \mathbf{L})^{\mathrm{T}} + \mathbf{L}^2\times\mathbf{I}\right] \\ &+ \frac{\mathrm{e}^{\lambda}}{24}\left[(\mathbf{L} - \lambda\mathbf{I})\times\left(\mathbf{L}^{\mathrm{T}} - \lambda\mathbf{I}\right)^2 + (\mathbf{L} - \lambda\mathbf{I})^2\times\left(\mathbf{L}^{\mathrm{T}} - \lambda\mathbf{I}\right)\right] \\ &+ \frac{\mathrm{e}^{\lambda}}{120}(\mathbf{L} - \lambda\mathbf{I})^2\times\left(\mathbf{L}^{\mathrm{T}} - \lambda\mathbf{I}\right)^2 + \mathrm{e}^{\lambda}(1 - \lambda)\mathscr{I} + \frac{1}{2}\mathrm{e}^{\lambda}(\mathbf{L}\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{L}),\end{aligned} \tag{73}$$

which is equivalent to (72) according to the identities (54) and (56).

Thus, under consideration of (67) the both relations (72) and (73) lead to the result

$$\begin{aligned}\exp(\mathbf{L}),_{\mathbf{L}} = &\mathscr{I} + \frac{1}{2}(\mathbf{L}\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{L}) + \frac{1}{6}\mathbf{L}\otimes\mathbf{L} + \frac{1}{6}\left(\mathbf{L}^2\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{L}^2\right) + \frac{1}{24}\left(\mathbf{L}^2\otimes\mathbf{L} + \mathbf{L}\otimes\mathbf{L}^2\right) \\ &+ \frac{1}{120}\mathbf{L}^2\otimes\mathbf{L}^2.\end{aligned} \tag{74}$$

On the other hand, the derivative of the exponential function can be obtained through the direct calculation of the power series (18). Under consideration of (8) this delivers

$$\exp(\mathbf{L}),_{\mathbf{L}} = \sum_{n=1}^{\infty}\frac{1}{n!}\sum_{k=0}^{n-1}\mathbf{L}^{n-1-k}\otimes\mathbf{L}^k. \tag{75}$$

Taking into account the relation (71) specific for the tensor $\mathbf{L}$ (66) we immediately arrive at (74).

8. Conclusion

Non-symmetric tensor power series and their derivative cannot generally be represented in a closed form by means of the well-known solutions for symmetric isotropic tensor functions based on the spectral decomposition. The problem is that non-symmetric tensor arguments with repeated eigenvalues can possess no spectral decomposition (in diagonal form) which necessitates to seek for other approaches. As such, we derived in the present paper a closed-form representation for tensor power series and their derivative. This representation is given in terms of the eigenvalues of the tensor arguments and is valid for all second-order tensors, symmetric and non-symmetric, with or without spectral decomposition. For the derivative of tensor power series two alternative closed-form solutions are proposed. One of these solutions is based on the definition of an isotropic tensor function without any reference to power series and can be of advantage (see Remark 4.1). Establishing some universal tensor identities connecting fourth-order tensors constructed with the tensor products "$\times$" and "$\otimes$" these two solutions are shown to be equivalent. It is interesting to note that one of these tensor identities represents the Rivlin's identity (Rivlin, 1955) written in terms of fourth-order tensors. In the special case of a tensor argument with a spectral decomposition our solutions reduce to the well-known result for symmetric isotropic tensor function (see e.g. Carlson and Hoger, 1986). Finally, we have illustrated the application of our closed formulas by an example being of special importance in large strain anisotropic elasto-plasticity. As such, we have considered the exponential function of the velocity gradient under simple shear. In this loading case the velocity gradient has a triple eigenvalue and only two linearly independent eigenvectors excluding the application of any other solutions based on the spectral decomposition. The results obtained by our closed-form representations coincide with those ones due to the direct calculation of infinite tensor power series.

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